Solving field equations in spinor electrodynamics.

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Abstract

Solutions of classical and quantum equations of motion in spinor electrodynamics are constructed within the context of perturbation theory. The solutions possess a graphical representation in terms of diagrams.

1   Introduction

Quantum spinor electrodynamics describes photons interacting with electrons and positrons. Scattering amplitudes of these interactions can be computed using pertubation theory. Terms of the corresponding series are represented by the Feynman diagrams.

The aim of this paper is to demonstrate a similar method for solving equations of motion in spinor electrodynamics.

Let $x = (x^0, x^1, x^2, x^3)$ be space-time coordinates,$\eta^{\mu\nu} = diag(1, -1, -1, -1)$ the metric tensor, and $\Box = \partial^\mu \partial_\mu$. The basic equations of spinor electrodynamics are defined by [1],

$$\Box A_\mu + j_\mu = 0,$$
$$i\bar\psi\gamma^\mu \overleftarrow{\partial}_\mu + e\bar\psi\gamma^\mu A_\mu + m\bar\psi = 0, \qquad (1)$$
$$i\gamma^\mu \partial_\mu \psi - e\gamma^\mu A_\mu \psi - m\psi = 0.$$

Here $A_\mu(x)$ is the four-vector potential of the electromagnetic field, $\psi(x)$ is the Dirac field with charge e and mass m, $\bar\psi(x)$ is the Dirac adjoint, γ^μ are

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the gamma matrices, $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$, and

$$j_\mu = \frac{e}{2}[\bar\psi, \gamma_\mu \psi].$$

The fields $A_\mu, \psi, \bar\psi$ are operator-valued functions in quantum theory and elements of the Berezin algebra in classical one [2]. The classical equations are compatible with the Lorenz gauge condition

$$\partial_\mu A^\mu = 0.$$

In what follows A_μ, ψ and $\bar\psi$ can be treated as elements of an arbitrary algebra over $\mathbb{C}$.

In the present paper we find explicit expressions for the electromagnetic and Dirac fields in terms of the corresponding free fields. These expressions are given by power series in charge e. Each term of the series is represented by a sum of diagrams. We ignore the divergences of quantum electrodynamics. The form of our solution in quantum case is different from that of ref. [3].

The paper is organized as follows. In the next section we review properties of the system of equations (1) and rewrite it in a convenient form. In Sec.3 we construct a solution for a wide class of nonlinear equations with quadratic nonlinearity and introduce a graphical representation. In Sec.4 we describe the solution of equation (1).

2 Equations of motion in integral form

Equations (1) can be transformed to their integral forms, see [4]. Suppose $A_{0\mu}, \psi_0$ and $\bar\psi_0$ are general solutions of the free equations

$$\Box A_{0\mu} = 0, \quad i\bar\psi_0 \gamma^\mu \overleftarrow{\partial}_\mu + m\bar\psi_0 = 0, \quad i\gamma^\mu \partial_\mu \psi_0 - m\psi_0 = 0.$$

Then equations (1) take the form

$$A = A_0 + \langle \bar\psi, \psi \rangle,$$
$$\bar\psi = \bar\psi_0 + \langle \bar\psi, A \rangle, \quad \psi = \psi_0 + \langle A, \psi \rangle. \tag{2}$$

Here $A = A_\mu dx^\mu, A_0 = A_{0\mu} dx^\mu, \langle \bar\psi, \psi \rangle = J_\mu dx^\mu,$

$$J_\mu = -\frac{1}{4\pi} \int_0^t \tau d\tau \int_S j_\mu(t - \tau, x^1 + \tau\xi^1, x^2 + \tau\xi^2, x^3 + \tau\xi^3) d\sigma_\xi,$$

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ξ^1, ξ^2 and ξ^3 are coordinates on the unit sphere S, σ_ξ is the area element on S, $t = x^0$,

$$\langle A, \psi \rangle = ie\gamma^0 e^{tK} \int_0^t e^{-tK}(\gamma^\mu A_\mu \psi)dt, \qquad K = \gamma^0(\sum_{k=1}^3 \gamma^k \partial_k + im),$$

$$\langle \bar{\psi}, A \rangle = -ie \int_0^t (\bar{\psi}\gamma^\mu A_\mu)e^{-t\overleftarrow{K}}dt e^{t\overleftarrow{K}}\gamma^0, \qquad \overleftarrow{K} = (\sum_{k=1}^3 \gamma^k \overleftarrow{\partial}_k + im)\gamma^0.$$

Let $\mathcal{A}, \Psi$ and $\bar{\Psi}$ be spaces of 1-forms, the Dirac spinors and Dirac adjoints, respectively. Define

$$\langle \psi, A \rangle = \langle A, \psi \rangle, \quad \langle \psi, \bar{\psi} \rangle = \langle \bar{\psi}, \psi \rangle, \quad \langle A, \bar{\psi} \rangle = \langle \bar{\psi}, A \rangle,$$
$$\langle A, A \rangle = \langle \psi, \psi \rangle = \langle \bar{\psi}, \bar{\psi} \rangle = 0.$$

Then for $\Phi = A + \psi(x) + \bar{\psi}(x) \in \mathcal{A} \oplus \Psi \oplus \bar{\Psi}$ the system of equations (2) takes the form

$$\Phi = \Phi_0 + \frac{1}{2}\langle \Phi, \Phi \rangle, \tag{3}$$

where $\Phi_0 = A_0 + \psi_0 + \bar{\psi}_0$.

One can make $\mathcal{A} \oplus \Psi \oplus \bar{\Psi}$ into a commutative algebra by taking as product the bracket $\langle \ , \ \rangle$. We have

$$\langle \mathcal{A}, \Psi \rangle \subset \Psi, \quad \langle A, \bar{\Psi} \rangle \subset \bar{\Psi}, \quad \langle \Psi, \bar{\Psi} \rangle \subset \mathcal{A},$$

all other brackets being zero.

3 Nonlinear equations with quadratic nonlinearity

Let $\mathcal{V}$ be a vector space and let

$$v = v_0 + \frac{1}{2}\langle v, v \rangle \tag{4}$$

be an equation, where $\langle \ , \ \rangle : \mathcal{V}^2 \to \mathcal{V}$ is a bilinear symmetric function, v_0 is a given vector, v is an unknown one.

To solve this equation we introduce a family of functions

$$\langle \ldots \rangle : \mathcal{V}^m \to \mathcal{V}, \quad m \geq 2,$$

defined for $v_1, \ldots, v_m \in \mathcal{V}$ by

$$\langle v_1, \ldots, v_m \rangle = \frac{1}{2} \sum_{r=1}^{m-1} \sum_{1 \leq i_1 < \ldots < i_r \leq m} \langle \langle v_{i_1}, \ldots, v_{i_r} \rangle, \langle v_1, \ldots, \widehat{v}_{i_1}, \ldots, \widehat{v}_{i_r}, \ldots, v_m \rangle \rangle, \quad (5)$$

where $\langle v \rangle = v$, and $\widehat{v}$ means that v is omitted. Since the functions $\langle v_1, \ldots, v_m \rangle$ depend only on the lower order functions $\langle v_1, \ldots, v_k \rangle$ with $k < m$, the solution of the system of equation (5) can be constructed by induction. For $m = 2$ we have an identity, for $m = 3$

$$\langle v_1, v_2, v_3 \rangle = \langle \langle v_1, v_2 \rangle, v_3 \rangle + \langle \langle v_1, v_3 \rangle, v_2 \rangle + \langle \langle v_2, v_3 \rangle, v_1 \rangle. \quad (6)$$

It is easy to prove that $\langle v_1, \ldots, v_m \rangle$ is an $m-$ linear symmetric function.

For $m \geq 2, 1 \leq i, j \leq m$, let

$$P_{ij}^m : \mathcal{V}^m \to \mathcal{V}^{m-1}$$

be a function defined by

$$P_{ij}^m(v_1, \ldots, v_m) = (\langle v_i, v_j \rangle, v_1, \ldots, \widehat{v}_i, \ldots, \widehat{v}_j, \ldots, v_m). \quad (7)$$

If $v \in \mathcal{V}$ is given by

$$v = P_{12}^2 P_{i_{m-2}j_{m-2}}^3 \ldots P_{i_2 j_2}^{m-1} P_{i_1 j_1}^m (v_1, \ldots, v_m) \quad (8)$$

for some $(i_1 j_1), \ldots, (i_{m-2}j_{m-2})$, we say that v is a descendant of $(v_1, \ldots, v_m)$.

From (6) it follows that $\langle v_1, v_2, v_3 \rangle$ is given by the sum of all the descendants of its arguments. The same is true for $\langle v_1, v_2 \rangle$. Assume that $\langle v_1, \ldots, v_k \rangle$, $k < m$, is given by the sum of all the descendants of $(v_1, \ldots, v_k)$. Each descendant of $(v_1, \ldots, v_m)$ can be written as

$$\langle r(v_I), s(v_J) \rangle, \quad (9)$$

where $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ are increasing multi-indexes[1], $I \cup J = (1, \ldots, m)$, $r(v_I)$ and $s(v_J)$ are some descendants of $v_I = (v_{i_1}, \ldots, v_{i_k})$

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[1]The multi-index $I = (i_1, \ldots, i_n)$ is said to be increasing if $i_1 < \ldots < i_n$.

and $v_J = (v_{j_1}, \ldots, v_{j_l})$, respectively. It is easy to verify that $I \cap J = \emptyset$. Summing all the functions (9), we get the right-hand side of (5). We have thus proved that $\langle v_1, \ldots, v_m \rangle$ is given by the sum of all the descendants of $(v_1, \ldots, v_m)$.

Each descendant can be represented by a diagram. In this diagram an element of $\mathcal{V}$ is represented by the line segment ——. A product $\langle v_i, v_j \rangle$ is represended by the vertex joining the line segments for v_i, v_j and $\langle v_i, v_j \rangle$. The graph for $P_{ij}^m(v_1, \ldots, v_m)$ (7) is depicted in Figure 1. Here the points labeled by $1, \ldots, m$ represent the ends of the lines for $v_1, \ldots, v_m$. Using this prescription, one can consecutively draw the diagrams for $P_{i_1 j_1}^m(v_1, \ldots, v_m)$, $P_{i_2 j_2}^{m-1} P_{i_1 j_1}^m(v_1, \ldots, v_m)$, $\ldots, v$ (8).The diagram for v has $m-1$ vertices and $m+1$ external lines. The auxiliary points $1, \ldots, m$ are removed.
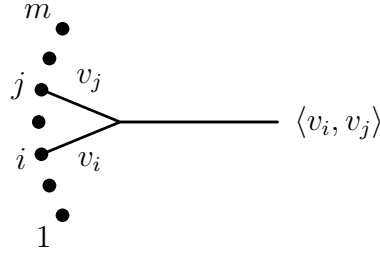

Figure 1. Diagram for $P_{ij}^m(v_1, \ldots, v_m)$.

For $v_1 = \ldots = v_m = v_0$ equation (5) takes the form

$$\langle v_0^m \rangle = \frac{1}{2} \sum_{r=1}^{m-1} \frac{m!}{r!(m-r)!} \langle \langle v_0^r \rangle, \langle v_0^{m-r} \rangle \rangle, \quad \langle v_0^r \rangle = \langle \underbrace{v_0, \ldots, v_0}_{r} \rangle. \quad (10)$$

We claim that

$$v = \langle e^{v_0} \rangle = \sum_{m=0}^{\infty} \frac{1}{m!} \langle v_0^m \rangle, \quad \langle v_0^0 \rangle = 0, \quad (11)$$

is a solution of equation (4). Indeed, substituting (11) in (4), we get

$$\sum_{m=2}^{\infty} \frac{1}{m!} \langle v_0^m \rangle = \frac{1}{2} \sum_{m=2}^{\infty} \sum_{r=1}^{m-1} \frac{1}{r!(m-r)!} \langle \langle v_0^r \rangle, \langle v_0^{m-r} \rangle \rangle.$$

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To conclude the proof, it remains to use (10).

For $\mathcal{V} = \mathbb{R}$, and $\langle v, v \rangle = v^2$ equation (4) takes the form $(1/2)v^2 - v + v_0 = 0$. From this it follows that $\langle e^{v_0} \rangle = 1 - \sqrt{1 - 2v_0}$, or equivalently, $\langle v_0^m \rangle = b_m v_0^m$, where $b_0 = 0, b_1 = 1, b_m = 1 \cdot 3 \cdot \ldots \cdot (2m - 3), m \geq 2$. Therefore the number of the descendants of $(v_1, \ldots, v_m)$ is b_m.

4   A solution of the electrodynamics equations

In spinor electrodynamics $\mathcal{V} = \mathcal{A} \oplus \Psi \oplus \bar{\Psi}$. Combining (3),(4) and (11), we get

$$A = P_{\mathcal{A}} \langle e^{A_0 + \psi_0 + \bar{\psi}_0} \rangle,$$

$$\psi = P_{\Psi} \langle e^{A_0 + \psi_0 + \bar{\psi}_0} \rangle, \qquad \bar{\psi} = P_{\bar{\Psi}} \langle e^{A_0 + \psi_0 + \bar{\psi}_0} \rangle, \tag{12}$$

where $P_{\mathcal{A}}$, P_{Ψ}, and $P_{\bar{\Psi}}$ are projectors on the spaces $\mathcal{A}, \Psi$, and $\bar{\Psi}$ respectively. The fields A_0, ψ_0 and $\bar{\psi}_0$ mutually commute under the bracket $\langle \ldots \rangle$, and therefore

$$\langle e^{A_0 + \psi_0 + \bar{\psi}_0} \rangle = \langle e^{A_0} e^{\psi_0} e^{\bar{\psi}_0} \rangle, \tag{13}$$

where

$$\langle e^{A_0} e^{\psi_0} e^{\bar{\psi}_0} \rangle = \sum_{p,q,r=0}^{\infty} \frac{1}{p!q!r!} \langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle.$$

The function $\langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle$ is defined as 0 if $p = q = r = 0$, and otherwise

$$\langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle = \langle \underbrace{A_0, \ldots, A_0}_{p}, \underbrace{\psi_0, \ldots, \psi_0}_{q}, \underbrace{\bar{\psi}_0, \ldots, \bar{\psi}_0}_{r} \rangle.$$

The system of equations (1) possesses an $U(1)$ global symmetry:

$$\psi(x) \to \psi'(x) = e^{i\theta} \psi(x), \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\theta} \bar{\psi}(x).$$

This implies

$$P_{\mathcal{A}} \langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle = \begin{cases} \langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle, & \text{if } q = r, \\ 0, & \text{otherwise}, \end{cases}$$

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$$
P_\Psi \langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle = \begin{cases} \langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle, & \text{if } q = r + 1, \\ 0, & \text{otherwise,} \end{cases}
$$

$$
P_{\bar{\Psi}} \langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle = \begin{cases} \langle A_0^p, \psi_0^q, \bar{\psi}_0^r \rangle, & \text{if } r = q + 1, \\ 0, & \text{otherwise.} \end{cases}
$$

Using these relations, (12) and (13), we get

$$
A = \sum_{p,q=0}^{\infty} \frac{1}{p!(q!)^2} \langle A_0^p, \psi_0^q, \bar{\psi}_0^q \rangle,
$$

$$
\psi = \sum_{p,q=0}^{\infty} \frac{1}{p!(q+1)!q!} \langle A_0^p, \psi_0^{q+1}, \bar{\psi}_0^q \rangle,
$$

$$
\bar{\psi} = \sum_{p,q=0}^{\infty} \frac{1}{p!q!((q+1)!} \langle A_0^p, \psi_0^q, \bar{\psi}_0^{q+1} \rangle.
$$

Drawing diagrams in electrodynamics, we depict the lines accociated with elements of $\mathcal{A}, \Psi$, and $\bar{\Psi}$ by $\sim\!\!\sim\!\!\sim$, $\longrightarrow$ and $\longleftarrow$, respectively. In Figure 2 we show the diagram for $P_{ij}^m(v_1, \ldots, v_m)$, where $v_i = \psi_i \in \Psi, v_j = \bar{\psi}_j \in \bar{\Psi}$.
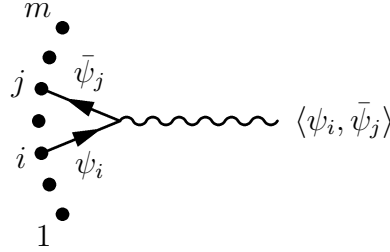

Figure 2. Diagram for $P_{ij}^m(v_1, \ldots, \psi_i, \ldots, \bar{\psi}_j, \ldots, v_m)$.

For example, the $O(e^2)$ contribution in A is given by

$$
\langle A_0, \psi_0, \bar{\psi}_0 \rangle = \langle \langle A_0, \psi_0 \rangle, \bar{\psi}_0 \rangle + \langle \langle A_0, \bar{\psi}_0 \rangle, \psi_0 \rangle.
$$

Recall that the product $\langle \ , \ \rangle$ includes a factor of e. The diagram for $\langle \langle A_0, \psi_0 \rangle, \bar{\psi}_0 \rangle$ is depicted in Figure 3.
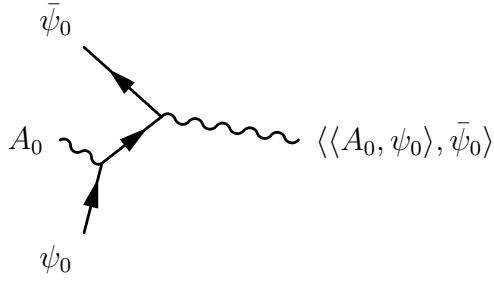
Figure 3. Diagram for $\langle\langle A_0, \psi_0\rangle, \bar{\psi}_0\rangle$

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